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THE SOLUTION OF STATISTICAL RADIATION TRANSPORT PROBLEMS BY THE MONTE-CARLO METHOD*

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A method is proposed for constructing statistical estimates for the characteristics of radiation described by a stochastic transport equation in media with relatively small fluctuations of the attenuation and scattering indices and an example of its application is introduced.

1. To describe the dispersion of light in turbid media the radiation transport equation (r.t.e.) is widely used /1/. However in many problems of atmospheric optics, the optics of a photographic layer etc., the maximum scale of the statistical non-homogeneity (macroscale) of the medium cannot be considered as negligibly small and the r.t.e. coefficients become fluctuating. To find the statistical characteristics of radiation in such a situation in /2/ and /3/ a direct computer modelling of the realizations of the medium is used with the last r.t.e. solution for each of them. Other methods include analytic averaging over the ensemble of realizations but these have narrower limits of applicability. Thus, the Markov approximation /4/ supposes that the small-angled r.t.e. is true and the macroscale is much shorter than the mean free path of the photon. In a number of papers (see the bibliography in /5/) the Markov nature of the medium along the trajectory of the photons is postulated to obtain closed equations. In /6/ the perturbation theory is used but the final formulae turn out to be too complex for calculations, as was noted in /7/. The method proposed in /7/ requires a knowledge of the conditions of the multipoint moments of the fluctuating parameters and its simplification assumes as was shown in /8/, the homogeneity of the medium in each of its realizations.

A method more successfully applied for a number of years has been the method based on the representation of fluctuations as a linear combination of certain functions with least random coefficients, /1/ and /8/. In /9/ a simple perturbation method was proposed that also used the triviality of the fluctuations but did not require their representation in a specific form. This method was taken as the basis for working out a method for the numerical evaluation of the statistical characteristics of radiation.

2. We will write the r.t.e. for the brightness I in the form /1/ and /9/

$$(\mathbf{s}\cdot\nabla+\sigma(\mathbf{r}))I(\mathbf{r},\mathbf{s})=\int_{4\pi}\sigma_s(\mathbf{r},\mathbf{s}\cdot\mathbf{s}')I(\mathbf{r},\mathbf{s}')\,d\mathbf{s}'+\Phi_0(\mathbf{r},\mathbf{s}). \tag{1}$$

Let σ denote the non-random parts of the extinction coefficient and σ_s the scattering coefficient. The corresponding non-perturbed values of the functionals are denoted by a bar, and the fluctuations by a tilde, for example $\sigma=\bar{\sigma}+\tilde{\sigma}$. We introduce the related fluctuations $\chi=\tilde{\sigma}/\bar{\sigma}$, $\chi_s=\tilde{\sigma}_s/\bar{\sigma}$ and the stochastic perturbation operator K:

$$(Kf)(\mathbf{r},\mathbf{s})=\int_{4\pi}\chi_s(\mathbf{r},\mathbf{s}\cdot\mathbf{s}')f(\mathbf{r},\mathbf{s}')\,d\mathbf{s}'-\chi(\mathbf{r})f(\mathbf{r},\mathbf{s}). \tag{2}$$

Suppose G is an operator whose kernel is Green's perturbation function of the r.t.e.. By formally considering the terms with $\tilde{\sigma}$ and $\tilde{\sigma}_s$ in (1) as sources the iterational formula can easily be obtained /9/

$$I=\sum_{n=0}^{\infty}(GK)^nI\approx I+GKI \tag{3}$$

here I is the solution of the initial r.t.e., the random function KI describes the local perturbation brightness, and GKI describes the effect of this perturbation on the brightness at other points. Higher terms of the series take into account the effects of the mutual influence of the fluctuations. In /9/ the sufficient conditions for correctness are obtained (3). By multiplying (3) and then averaging we obtain the brightness moments. Thus, if $\langle K\rangle=0$ (the angle brackets indicate averaging over the aggregate of the realizations σ, σ_s), then at a lower order of perturbation theory for the correlation function $D_{12}=\langle I_1I_2\rangle-\langle I_1\rangle\langle I_2\rangle$ we obtain

$$D_{12}=G_{13}G_{24}\langle(KI)_3(KI)_4\rangle=G_{13}G_{24}\langle K_{35}K_{46}\rangle I_5I_6, \tag{4}$$

where the indices j correspond to points of the phase space $x_j=(\mathbf{r}_j,\mathbf{s}_j)$ and the integration that is implied by the operators K and G is carried out over variables with repeating indices.

3. In the final formulae (type (4)) Green's function of the r.t.e. is introduced that, in practice, as a rule, is not known. The proposed method of calculation enables this difficulty to be avoided. It is based on the fact that since Gf has the meaning of the brightness, generated by the volume sources f, whose calculation by the Monte Carlo method is

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well-known, it is possible to model the corresponding trajectory instead of distinctly finding the kernel of each operator G and subsequent integration. We shall clarify this by the example of a functional (φ_{12}, D_{12}), by considering for simplicity that the scattering and absorption fluctuations are single-valuedly connected by the relations

$$\chi_s(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}') = \chi_1(\mathbf{r})\, b(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}'), \qquad \chi(\mathbf{r}) = \left[\int_{4\pi} b(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}')\, d\mathbf{s}' - 1\right]\chi_1(\mathbf{r}), \tag{5}$$

where $b(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}')$ is a deterministic function. Then from (2) it follows that

$$K = \chi_1 K^{(0)}, \qquad (K^{(0)}f)(x) = \int k^{(0)}(x, x')f(x')\, dx',$$

where

$$k^{(0)}(x, x') = \delta^3(r - r')\left[b(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}') + \delta_0(\mathbf{s}, \mathbf{s}')\left(1 - \int_{4\pi} b(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}'')\, d\mathbf{s}''\right)\right],$$

and δ_0 is the delta function on the unit sphere.

Suppose $k(x, x')$ is the kernel of a non-perturbed integral r.t.e. /1, p.15/. We shall model two trajectories $\{x_m^i | m = 0, 1, \ldots, N_i\}$, $i = 1, 2$, with identical initial intensity $\rho(x^1, x^2)$, with transport intensities $r^i(x, x')$ and cut-off probability $p^i(x)$. By the standard method in /1/ it is easy to prove the unbiasedness of the following estimate "for absorptions":

$$\eta_{N_1N_2} = \psi(x_0^1, x_0^2)\,\rho^{-1}(x_0^1, x_0^2)\, Q_{N_1}^1 Q_{N_2}^2 \varphi(x_{N_1}^1, x_{N_2}^2), \tag{6}$$

where

$$Q_N^i = \frac{1}{P^i(x_N^i)} \prod_{j=1}^{N} \frac{k(x_{j-1}^i, x_j^i)}{r^i(x_{j-1}^i, x_j^i)\,[1 - p^i(x_{j-1}^i)]},$$

and of the corresponding estimate "for collisions":

$$\xi = \sum_{n=0}^{N_1}\sum_{m=0}^{N_2} \eta_{nm} p^1(x_n^1) p^2(x_m^2), \tag{7}$$

$$M\xi = M\eta = (\varphi_{12}, G_{13}G_{24}\psi_{34}) = (\varphi_{12}, D_{12}).$$

Here, compared with (4) and (5)

$$\psi(x, x') = (K^{(0)}I)(x)\,(K^{(0)}I)(x')\,C(x, x'), \tag{8}$$

and $C(x, x') = \langle \chi_1(\mathbf{r})\chi_1(\mathbf{r}')\rangle$ is the correlation inhomogeneity function, assumed known.

4. If there is no explicit formula for solving I of the initial r.t.e. then in (6) it is possible, in turn, to replace the function ψ by its statistical value. For this we shall construct two more trajectories $(\rho^i(x), r^i(x, x'), p^i(x)$, $i = 3, 4)$ according to which we shall evaluate the quantities $I = G\Phi$ in formula (8) ($\Phi = \Phi_0/\bar{\sigma}$ is the brightness of the sources), after which we shall model the action $K^{(0)}$, selecting the transport intenstiy $r^{(0)}(x, x')$ is such a way that the relation $k^{(0)}/r^{(0)}$ is bounded. Suppose $\delta(x, x') = \delta^3(\mathbf{r} - \mathbf{r}')\delta_0(\mathbf{s}, \mathbf{s}')$. Taking (8) into account we obtain

$$\psi_{12} = (\delta_{13}\delta_{24}, \psi_{34}) = (C(x_3, x_4)\delta(x_3, x_1)\delta(x_4, x_2), (K^{(0)}G\Phi)(x_3)\,(K^{(0)}G\Phi)(x_4)).$$

The estimates of this functional for absorptions take in the following form:

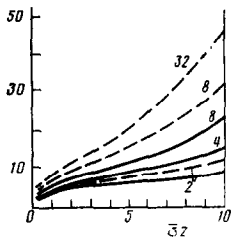
$$\eta_{N_3N_4}^{\psi} = \left[\prod_{i=3,4} \frac{\Phi(x_0^i)}{\rho^i(x_0^i)} Q_{N_i}^i \frac{k^{(0)}(x_{N_i}^i, x_i)}{r^{(0)}(x_{N_i}^i, x_i)}\right] C(x_3, x_4)\delta(x_1, x_3)\delta(x_2, x_4). \tag{9}$$

If now in (6) we choose $\rho_{12} = \delta_{13}\delta_{24}$, i.e. taking the first and second trajectories of the point of phase space as source points, obtained by transport with intensity $r^{(0)}$ from finite points of the third and fourth trajectories then the singularity of (9) after substitution into (6) contracts and we reach the final estimate of the functional (φ_{12}, D_{12}):

$$\eta = \left[\prod_{i=3,4} \frac{\Phi(x_0^i)}{\rho^i(x_0^i)} Q_{N_i}^i \frac{k^{(0)}(x_{N_i}^i, x_0^{i-3})}{r^{(0)}(x_{N_i}^i, x_0^{i-2})}\right] C(x_0^1, x_0^2) Q_{N_1}^1 Q_{N_2}^2 \varphi(x_{N_1}^1, x_{N_2}^2).$$

Analogous estimates for other functionals can be constructed in a similar way and also the conjugate local values of the brightness moments. Moreover, the estimate obtained can be modified using (7). Clearly, the next modification can be obtained if η is replaced by η^{ψ} in (7) and the value obtained for the function ψ is then used in (6), but this makes it necessary to construct a "second generation" trajectory (in the case under consideration the first and second) starting from each of the collision points on the "first generation" trajectories (the third and fourth). It would seem that the intermediate estimates proposed in /10/ are more convenient here.

5. As an example of the use of the proposed algorithm, results are presented in Fig.1 of a calculation on a BESM-6 computer of the distribution of the relative fluctuations in brightness averaged along the line $[D(z, z)/\langle\chi_1^2\rangle]^{1/2}/I(z)$, and, for a depth z of an isotropically scattering layer, the boundary $z = 0$ of which is uniformly illuminated. Parameters were taken that are characteristic for unexposed photosensitive layers: an optical thickness of 10,



Fresnel reflecting boundaries with a relative refractive index of 1.5, and a scattering albedo of 0.90 (the solid curves) and 0.95 (the dashed curves). The numbers on the right are the values of the correlation radius of the inhomogeneity on a scale of the mean free path of the photon $\bar{\sigma}^{-1}$. In the given example the function b in (5) is a constant and the dependence on its value lies within the limits of error (less than 10%).

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TWO-LAYER COMPLETELY CONSERVATIVE DIFFERENCE SCHEMES FOR THE EQUATIONS OF GAS DYNAMICS IN EULER VARIABLES*

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A family of two-layer completely conservative difference schemes is constructed for the spatially one-dimensional equations of gas dynamics. It is possible to obtain complete conservativeness of the difference scheme by profiling the time weights in a space matched to the solution of the problem.

1. Various semi-empirical principles play an important role when constructing effective difference schemes for the numerical solution of non-linear problems in mathematical physics. These include, in particular, the principle of complete conservativeness /1, 2/, which is well recommended in practice: in a difference scheme describing the behaviour of the discrete model of the object, in addition to analogues of the main laws of conservation (as in classical conservative schemes), certain additional relations of a physical nature must be satisfied (in gas dynamics, for example, the balance of individual forms of energy).

Completely conservative difference schemes that are two-layer with respect to time, have been constructed for the non-stationary equations of gas dynamcis (both one-dimensional and two-dimensional) using Lagrangian coordinates /1, 3-5/. Attempts to construct similar schemes

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