On exact solutions to nonlinear dissipative equations

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An approach is proposed to construct some exact explicit travelling wave (TW) solutions of nonlinear dispersive equations with dissipation, which seems not to be integrable by means of the inverse scattering transform method.

In one of his basic papers in 1883 Sophus Lie considered an integration problem for a general nonlinear o.d.e. that can be written in the form:

$$y'' + A(x,y)(y')^3 + B(x,y)(y')^2 + C(x,y)y' + D(x,y) = 0 \tag{1}$$

and referred to as the Lie equation (LE). Reduction of it to the 1st order o.d.e. was widely investigated and recently the substitution was found in [3] to reduce an autonomous form of LE to the 1st order Abel equation. In particular case for $A = B = 0$, that is typical for simplest nonlinear reaction-diffusion problem, we proved recently that an autonomous version of (1) with $C = C(y)$, $D = D(y)$ has a general exact solution in implicit form. An explicit solution occurs when an inversion problem for an Abelian integral can be solved, and several solutions were found.

Various wave propagation problems can be reduced to the problem on self-similar solutions to an autonomous equation, containing the nonlinear, dispersive and dissipative terms, and, consequently, to the equation for nonlinear oscillator with damping, written for the unknown $u(z)$:

$$u'' = a(u)u' + b(u) + c(u)(u')^2 + d(u)(u')^3, \tag{2}$$

Here $' \equiv d/dz, z = x \pm Vt$, $b(u)$ is a nonlinearity function, $a(u), c(u)$ and $d(u)$ are functions of dissipation. In physics the equation (2), containing both polynomials $b(u) = \sum\ b_i u^i$ and $a(u) = \sum\ a_i u^i$, corresponds to the TW problem for the KdV-Burgers' equation ($i = 2$, $\beta = 0$,), or for the modified KdV-B equation ($i = 3$, $\alpha = 0$). Various nonlinear equations are reducible to (2), e.g., the equation for nonlinear dissipative waves in solid wave guides , where a_0, c_1, c_2 - constants), the Fisher equation ($i = 2$, $b_2 = -b_1$), the Ginzburg-Landau equation ($b_2 = 0$), the nonlinear reaction-diffusion (NRD) equation and many others.

We found a new substitution and proved that *the generalized autonomous Lie equation of 2nd order, (2), containing autonomous* **arbitrary** *both nonlinear and dissipative terms, can be reduced to the Abel equation of the 1st order:* $v' = b(u)v^3 + F(u)$, the non-autonomous one with cubic nonlinearity. This result is quite general, applicable also to find TW solutions to various initial problems, described by means of different nonlinear equations.

The Abel equation is not integrable in general case, and we find the set of additional integrability conditions, and the closed form solutions to (2) having polynomial nonlinearities, in terms of rational (or even polynomial) functions R and S with respect to the Weierstrass function $\wp(z+c)$ or $\wp(\exp(z+c))$ as $F(z) =$R$(\wp(z+c))+$S$[\wp(z+c)]\wp'(z+c)$. When $d(u)=0$, (2), the explicit solutions for $n=2$ and $n=3$ are obtained.

Various problems of nonlinear wave or pattern formation in molecular biology, kinetics of thin film growth, nanostructure technology, heat transfer in nonlinear medium etc. can be reduced to nonlinear p.d.e. with dissipative or diffusive terms. Exact solutions to almost all of them were not available recently.

New substitution was found [1]to reduce a problem of quasistationary solution of the NRD equation with *arbitrary* diffusion D and reaction A,

$$u_t = (D(u)u_x)_x + A(u)$$

that is, of the o.d.e. of the form:

$$(D(u)u')' + A(u) = 0$$

written with respect to an unknown $u(\cdot)$ to the *linear* o.d.e. of the 1st order. Some exact and explicit solutions are found for these equations in closed form in terms of the Weierstrass elliptic function $\wp$, containing kink or soliton solutions as appropriate limits. Additional conditions for the equation coefficients and wave parameters are found for the existence of these solutions, [2], [4].

Tentative applications to the problem on kinetics of thin film growth (or of wire-like nanostructures) are discussed.

References

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